ILLC PhD Pilot Study

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1 Description of the topic and background

This is a proposal for a thesis in the interface of descriptive set theory and computable analysis, in a continuation of my MoL thesis [9]. There, I was interested in studying connections between game characterizations of classes of functions in descriptive set theory and the theory of Weihrauch reducibility, in particular searching for counterparts in computable analysis to the game characterizations obtained by Brian Semmes in his PhD thesis [25] at the ILLC, also under the supervision of Benedikt Löwe.

Function classes in descriptive set theory

We are interested in game characterizations of classes of functions in the Baire space ω^ω, following a well-established tradition in the area dating back to the work of Wadge [27] and with developments by Van Wesep, Andretta, Motto Ros, and Semmes, among others (see, e.g., [9] and the references therein).

Characterizing a function class by a game can help solve problems for that class which could otherwise be harder. Take, for example, the situation with the famous Jayne-Rogers theorem:

The Jayne-Rogers Theorem ([12]). *If $f : \omega^\omega \longrightarrow \omega^\omega$, then the preimages of $\mathbf{\Sigma}^0_2$ sets under f are $\mathbf{\Sigma}^0_2$ sets if, and only if, there is a partition of ω^ω into countably many closed sets, such that f restricted to each part is continuous.*

One of the most challenging problems in the area is to prove generalizations of the Jayne-Rogers theorem — see, e.g., [18, 23, 26] and references therein for an up-to-date overview of the situation.

For reasons which will become clear shortly, let us refer to the class of functions defined by the partition property mentioned in the Jayne-Rogers theorem by $\mathbf{\Lambda}^{\mathsf{P}}_{2,2}$. This class has been characterized by a game, as follows. The *backtrack game* for a function f is played between two players, **I** and **II**, in ω rounds. At each round, **I** plays a natural number, and **II** responds by either playing a natural number, passing, or erasing all of her past moves (*backtracking*), as long as in the long run she (1) plays natural numbers infinitely often, and (2) only decides to backtrack a finite number of times. Therefore, in the long run **I** and **II** determine elements x and y of ω^ω, respectively, and **II** wins that run of the game exactly when $f(x) = y$.

Andretta proved in [1] that this game characterizes $\mathbf{\Lambda}^{\mathsf{P}}_{2,2}$ in the sense that $f \in \mathbf{\Lambda}^{\mathsf{P}}_{2,2}$ iff **II** has a winning strategy in the backtrack game for f. Of course, by the Jayne-Rogers theorem this game then characterizes the class of functions preserving $\mathbf{\Sigma}^0_2$ under preimages.

In his PhD thesis [25] at the ILLC, Semmes was able to give a new proof of the Jayne-Rogers theorem by closely analyzing the backtrack game. Furthermore, he was able to generalize the Jayne-Rogers theorem to two higher levels of a certain hierarchy of classes of functions (to be defined below) by using new games he defined.

This hierarchy of classes of functions is defined as follows. Given countable ordinals $0 < \alpha \leq \beta$, we denote by $\mathbf{\Lambda}_{\alpha,\beta}$ the class of functions $f : \omega^\omega \longrightarrow \omega^\omega$ such that the preimage of any $\mathbf{\Sigma}^0_\alpha$ set under f is a $\mathbf{\Sigma}^0_\beta$ set. Thus $\mathbf{\Lambda}_{1,1}$ is the class of continuous functions, $\mathbf{\Lambda}_{1,\alpha+1}$ is the α^{th} Baire class, and $\mathbf{\Lambda}_{2,2}$ is the class of functions characterized in the Jayne-Rogers theorem.

We can also define generalized classes of functions defined by partition properties as follows. First, given a class Γ of subsets of ω^ω and a class Λ of functions in ω^ω, we say that a function $f : \omega^\omega \longrightarrow \omega^\omega$ is *piecewise* Λ *on a* Γ *partition* when there exists a partition of ω^ω into countably many sets which are in Γ, and such that f restricted to each part is in Λ. Now, given natural numbers $0 \leq n \leq m$, we denote by $\mathbf{\Lambda}^{\mathsf{P}}_{n+1,m+1}$ the class of functions which are piecewise Baire class $m - n$ on a $\mathbf{\Pi}^0_m$ partition.

We can then state one version of the conjectured generalization of the Jayne-Rogers theorem, usually attributed to Andretta [2], as follows.

Conjecture 1 (Jayne-Rogers for finite levels)**.** $\mathbf{\Lambda}_{n+1,m+1} = \mathbf{\Lambda}^{\mathsf{P}}_{n+1,m+1}$ *holds for all natural numbers* $n \leq m$.

The results mentioned above can now be stated in a precise way.

Theorem 2 (Jayne-Rogers, Semmes)**.** *1.* $\mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}^{\mathsf{P}}_{2,2}$*;*

2. $\mathbf{\Lambda}_{2,3} = \mathbf{\Lambda}^{\mathsf{P}}_{2,3}$*;*

3. $\mathbf{\Lambda}_{3,3} = \mathbf{\Lambda}^{\mathsf{P}}_{3,3}$*.*

To the best of our knowledge, the remaining cases of Conjecture 1 remain open. However, we already know that a direct generalization of Conjecture 1 cannot hold in general for infinite levels — there exist, for example, functions of Baire class 1 which are not piecewise Baire class 0 *at all* (i.e., with partitions of any complexity); since Baire class 1 functions must pull back $\mathbf{\Sigma}^0_\omega$ to $\mathbf{\Sigma}^0_\omega$ under preimages, it cannot be the case that $\mathbf{\Lambda}_{\omega,\omega}$ is the class of functions which are piecewise Baire class 0 on some partition. See, e.g., [7, 18, 23] for several precise counterexamples and deeper discussion.

A better conjecture in the general case is given by Kihara [14] as follows. Given countable ordinals $\alpha \leq \beta$, we denote by $\beta \leftharpoondown \alpha$ the least ordinal γ such that

$$\gamma + 1 + \alpha > \beta.$$

Thus $m \leftharpoonup n = m - n$ for finite $n \leq m$, but e.g. $\omega \leftharpoonup \omega = \omega$. We now define $\mathbf{\Lambda}^{\mathsf{P}}_{\alpha+1,\beta+1}$ to be the class of functions $f : \omega^\omega \longrightarrow \omega^\omega$ which are piecewise Baire class $\beta \leftharpoonup \alpha$ on a $\mathbf{\Pi}^0_\beta$ partition.

Conjecture 3 (Generalized Jayne-Rogers)**.** *$\mathbf{\Lambda}_{\alpha+1,\beta+1} = \mathbf{\Lambda}^{\mathsf{P}}_{\alpha+1,\beta+1}$ holds for all countable ordinals $\alpha \leq \beta$.*

Although some positive partial results are given by Kihara in [14] using tools from computable analysis, to the best of our knowledge Conjecture 3 remains open for all cases not included in Theorem 2.

Computable analysis and the Weihrauch reducibility relation

We will only give an outline of some aspects of this area here; for a more thorough introduction we refer the reader to [20, 28].

Computable analysis is an area of mathematics concerned with the computational content of theorems from mathematical analysis. These mathematical theorems are treated as the objects of the theory, and there is a notion of reducibility — the *Weihrauch reducibility* — which allows us to classify these theorems in terms of their computational content, in a similar way to what is done with the reducibility relations usually studied in computability theory. We restrict ourselves to theorems of $\forall\exists$ form, such as the intermediate value theorem

"For all continuous $f : [0,1] \longrightarrow \mathbb{R}$ such that $f(0) \cdot f(1) < 0$,
there exists $x \in (0,1)$ satisfying $f(x) = 0$",

which are then seen as relations between the spaces over which the universal and existential quantifiers range, respectively. The goal is to precisely quantify how much computational power one needs in order to transform (some precise description of) an element in the domain of the relation into (some precise description of) an element in the codomain related to it. Note that in order to properly understand the concepts and results in computable analysis, it is helpful to think of a relation as a *non-deterministic function*, so that when computing a relation as described above, the task is merely to compute *some* element related to the given input, and it is irrelevant *which* such element is computed.

The Weihrauch reducibility relation encapsulates the intuitive idea that a relation R is reducible to a relation S when being able to compute S already implies being able to compute R. Since the objects of study in analysis are mostly infinitary in nature, this necessarily involves a more general notion of computation than the usual one. However, since we are mostly interested in a certain *topological* variant of this theory, where notions such as computable functions, recursively enumerable sets, etc., are replaced by their respective topological counterparts of continuous functions, open sets, etc., we will not go into any details about this general notion of computation here. It remains true that we will need tools that allow us to talk about topological aspects of certain generalized spaces of objects, in particular ones that don't carry a natural topology with them. We do this by transferring notions from the well-studied Baire space ω^ω to these spaces via *coding*, as follows.

A *represented space* is a pair $\mathbb{X} = (X, \delta_X)$ where $\delta_X : \omega^\omega \longrightarrow X$ is surjective and may be partial. If $\delta_X(p) = x$, then we say that p is a δ_X*-name* of x. We will always assume that ω^ω is represented by the identity function id, and that ω is represented by the function $\delta_\omega : p \mapsto p(0)$.

A (partial) function $f : \omega^\omega \longrightarrow \omega^\omega$ is a *realizer* for a relation $R \subseteq \mathbb{X} \times \mathbb{Y}$, denoted $f \vdash R$, when $\mathrm{dom}(f)$ contains all the δ_X-names of elements in $\mathrm{dom}(R)$, and

$$f(p) \text{ is a } \delta_Y\text{-name of an element of } R(x)$$

whenever

$$p \text{ is a } \delta_X\text{-name of } x \in \mathrm{dom}\, R,$$

where we use the usual notation $R(x) := \{y \in Y\,;\, xRy\}$. Thus $f \vdash R$ when f is a deterministic simulation of R which works on the codes of the elements of the spaces $\mathbb{X}$ and $\mathbb{Y}$.

Finally, we say $R \subseteq \mathbb{X} \times \mathbb{Y}$ is *Weihrauch reducible* to $S \subseteq \mathbb{U} \times \mathbb{V}$, denoted by $R \leq_{\mathsf{W}} S$, when there exist continuous (partial) functions $\textsc{in}, \textsc{out} : \omega^\omega \longrightarrow \omega^\omega$ such that

$$\textsc{out}(\mathrm{id}, g \circ \textsc{in}) \vdash R$$

whenever $g \vdash S$.

Therefore, intuitively we have $R \leq_{\mathsf{W}} S$ when the problem of computing R can be transformed into the problem of computing S by just continuously packing and unpacking the information contained in the input and in the output of the computation of S on the re-packed input, respectively. For technical reasons we also allow the unpacking function access to the original input; this is done by using a suitable pairing function for ω^ω in order to pass two arguments to $\textsc{out}$ instead of just one.

Our main interest with Weihrauch reducibility is that it provides us with another way of characterizing classes of functions in ω^ω by looking for *complete* relations for that class. More precisely, given a class $\mathcal{C}$ of functions in ω^ω we can try to find a relation R between represented spaces $\mathbb{X}$ and $\mathbb{Y}$ with the property that, for any $f : \omega^\omega \longrightarrow \omega^\omega$, we have

$$f \in \mathcal{C} \iff f \leq_{\mathsf{W}} R.$$

For the cases that interest us, these relations have generically been referred to as *choice principles* in the literature. Two examples of choice principles characterizing classes of functions are the following.

Theorem 4 (Brattka [4]). *For each $k \in \omega$, the principle of k-*countable choice *$\mathsf{C}_k : \omega^\omega \longrightarrow \omega^\omega$ given by*

$$\mathsf{C}_k(x)(n) = \begin{cases} 0, & \text{if } \exists n_{k-1} \forall n_{k-2} \cdots Q n_0.\ (x)_n(\ulcorner n_k, n_{k-1}, \ldots, n_0 \urcorner) \neq 0 \\ 1, & \text{otherwise,} \end{cases}$$

where the $\exists$ and $\forall$ quantifiers alternate (thus Q is either $\exists$ or $\forall$ depending on the parity of k), is Weihrauch-complete for the Baire class k functions.

Here $\ulcorner \cdot \urcorner$ is some suitable $(k+1)$-ary tupling function on the natural numbers.

Theorem 5 (Brattka, de Brecht, Pauly [5])**.** *The principle of* discrete choice $\mathsf{C}_\omega \subseteq \omega^\omega \times \omega$ *given by*

$$\mathsf{C}_\omega(x) = \omega \smallsetminus \operatorname{ran}(x)$$

is Weihrauch-complete for $\mathbf{\Lambda}^{\mathsf{P}}_{2,2}$.

Let us close this section with a concrete example of how finding a complete choice principle for a class may aid in characterizing that class by a game, by considering the backtrack game.

First, it is easy to use the backtrack game to define a realizer g for C_ω: at each round, **II** just plays the least natural number that hasn't been played by **I**, backtracking whenever this changes. Assuming $x \in \operatorname{dom}(\mathsf{C}_\omega)$, this strategy will only tell **II** to backtrack finitely many times, so this is a valid strategy for **II** in the backtrack game. Therefore, it is easy to see that, if $f \in \mathbf{\Lambda}^{\mathsf{P}}_{2,2}$, then by Theorem 5 we have $f \leq_{\mathsf{W}} \mathsf{C}_\omega$, which in turn easily implies $f \leq_{\mathsf{W}} g$. But this implies the existence of a winning strategy for **II** in the backtrack game for f — **II** just needs to simulate a run of the backtrack game for g where **I** plays $\textsc{in}(x)$ instead of x and **II** follows the strategy given above,, and use $\textsc{out}$ to translate the resulting moves back to the game for f, where $\textsc{in}, \textsc{out}$ are the reducing functions witnessing $f \leq_{\mathsf{W}} g$.

Conversely, if we have a winning strategy for **II** in the backtrack game for some function f, then we can easily define a continuous function $\textsc{in}$ with the property that $n \notin \operatorname{ran}(\textsc{in}(x))$ if, and only if, **II** does not backtrack after round n when she follows her winning strategy and **I** plays x, i.e., iff at and after round n, none of **II**'s moves will be changed again. Then applying any realizer of C_ω to $\textsc{in}(x)$ will give us one of these rounds, and we can now easily define $\textsc{out}$ to simulate the run of the game, wait until the indicated round, and then simply copy the output. Since the strategy we started with was winning for **II**, we get

$$\textsc{out}(x, g \circ \textsc{in}(x)) = f(x),$$

for any $g \vdash \mathsf{C}_\omega$, i.e., $f \leq_{\mathsf{W}} \mathsf{C}_\omega$. Finally, by Theorem 5 this implies $f \in \mathbf{\Lambda}^{\mathsf{P}}_{2,2}$ as desired.

In my master's thesis at the ILLC [9], I studied and uncovered some connections between games for functions in descriptive set theory and Weihrauch reducibility, in the style of the one just described, including connections going in the converse direction, i.e., how to obtain some complete choice principles for a class given a game for the class with certain properties.

2 Work done so far

In these first 9 months of my PhD at the ILLC, aside from the usual obligations of a PhD candidate at this institute such as being a teaching assistant to Benedikt Löwe in *Axiomatic Set Theory*, organizing the *Amsterdam-Hamburg Set Theory Encounter* on April 24$^{\text{th}}$, and participating in the presentations course, my focus has not only been on studying the literature in depth (as is to be expected) and coming up with my first results (detailed below), but also strongly on forming a good network of contacts with researchers in the relevant areas, as well as with fellow PhD candidates at this and other institutions.

With this goal in mind, I have been to the following conferences during this time: *INFTY Final conference* (March), in Bonn, where I met with Luca Motto Ros and established contact with good prospects for future collaboration; *PhDs in Logic* (April), in Utrecht, which is a great opportunity to get to know other PhD candidates in logic, especially in the Netherlands and neighboring countries, and where I gave a talk about the topic of my research; *Young Set Theory Workshop* (May), in Będlewo, which is an annual event designed to foster contact between young researchers in set theory and where I met Kevin Fournier and Yann Pequignot, two students of Alain Louveau who have done some research in related areas and who were able to roughly sketch to me some of Louveau's unpublished work relating to Brian Semmes's games mentioned in the previous section; *Logic Colloquium* (July), in Vienna, where I met with Arno Pauly and established the first contacts of what eventually turned into a research visit to Cambridge (August); and *Colloquium Logicum* (September), where I again met with Arno Pauly and also with Vasco Brattka, and gave a talk about my first technical contributions — generalizations of Theorems 4 and 5 to higher Baire classes and partition classes, respectively (Theorems 7, 8, and 10 below).

Before stating my results in a precise way, let us recall the following well-known notion. We say that a set $X \subseteq \omega^\omega$ is *Wadge-complete* for a class Γ of subsets of ω^ω when

$$\Gamma = \{Y \subseteq \omega^\omega\,;\, Y \text{ is the preimage of } X \text{ under some continuous } f : \omega^\omega \longrightarrow \omega^\omega\}.$$

Theorem 6 (Wadge, see e.g. [13, Theorem 22.10])**.** *For every $\alpha < \omega_1$ there exist Wadge-complete sets for $\mathbf{\Sigma}^0_\alpha$; in fact, we have that a set is Wadge-complete for $\mathbf{\Sigma}^0_\alpha$ if, and only if, it is in $\mathbf{\Sigma}^0_\alpha$ but not in $\mathbf{\Pi}^0_\alpha$.*

Given $X \subseteq \omega^\omega$, we define $\mathsf{F}_X : \omega^\omega \longrightarrow \omega^\omega$ and $\mathsf{R}_X \subseteq \omega^\omega \times \omega$ by

$$\begin{aligned} \mathsf{F}_X(x)(n) &:= \begin{cases} 0, & \text{if } (x)_n \in X \\ 1, & \text{otherwise.} \end{cases} \\ \mathsf{R}_X(x) &:= \{n \in \omega\,;\, (x)_n \notin X\}. \end{aligned}$$

Theorem 7 (N., around May)**.** *For any $\alpha < \omega_1$, if X is Wadge-complete for $\mathbf{\Sigma}^0_\alpha$, then F_X is Weihrauch-complete for Baire class α.*

In particular, since the set

$$X_k = \{x \in \omega^\omega\,;\, \exists n_{k-1} \forall n_{k-2} \cdots Q n_0.\ x(\ulcorner n_{k-1}, \ldots, n_0 \urcorner) \neq 0\} \tag{1}$$

is Wadge-complete for $\mathbf{\Sigma}^0_k$ and $\mathsf{F}_{X_k} = \mathsf{C}_k$, we get Theorem 4 as a corollary.

Theorem 8 (N., around June)**.** *For any $\alpha < \omega_1$, if X is Wadge-complete for $\mathbf{\Sigma}^0_\alpha$, then R_X is Weihrauch-complete for the functions which are piecewise continuous on a $\mathbf{\Pi}^0_\alpha$ partition.*

Corollary 9. *If X is Wadge-complete for $\mathbf{\Sigma}^0_n$, then R_X is Weihrauch-complete for $\mathbf{\Lambda}^{\mathsf{P}}_{n+1,n+1}$.*

In particular, taking X_1 as defined in (1) above, we get that R_{X_1} is Weihrauch-complete for $\mathbf{\Lambda}^{\mathsf{P}}_{2,2}$, and since it is not hard to see that R_{X_1} is Weirauch-equivalent to C_ω, we get Theorem 5 as a corollary.

Finally, in [6], Brattka and Pauly introduce a composition of relations — and prove that it is indeed a well-defined notion — as follows. Given relations R and S between represented spaces, let

$$R \star S := \max_{\leq_{\mathsf{W}}}\{R' \circ S' \,;\, R' \leq_{\mathsf{W}} R \text{ and } S' \leq_{\mathsf{W}} S\}.$$

Theorem 10 (N., August, restating an earlier result after comments by Pauly)**.** *For any $\alpha, \beta < \omega_1$, if X is Wadge-complete for $\mathbf{\Sigma}^0_\alpha$ and Y is Wadge-complete for $\mathbf{\Sigma}^0_\beta$, then $\mathsf{F}_Y \star \mathsf{R}_X$ is Weihrauch-complete for the functions which are piecewise Baire class β on a $\mathbf{\Pi}^0_\alpha$ partition.*

Corollary 11. *For any $\alpha \leq \beta < \omega_1$, if X is Wadge-complete for $\mathbf{\Sigma}^0_\alpha$ and Y is Wadge-complete for $\mathbf{\Sigma}^0_{\beta \leftharpoondown \alpha}$, then $\mathsf{F}_Y \star \mathsf{R}_X$ is Weihrauch-complete for $\mathbf{\Lambda}^{\mathsf{P}}_{\alpha+1,\beta+1}$.*

More recently, having started by using the choice principles R_X as a guide in the sense described at the end of Section 1, I have defined an operation assigning to each game G and natural number n a game $(G)_n$ such that the following holds.

Theorem 12 (N., September and October)**.** *If G characterizes a class Λ of functions in ω^ω, then $(G)_n$ characterizes the functions which are piecewise Λ on a $\mathbf{\Pi}^0_n$ partition.*

Thus, applying this construction to the well-known *Wadge game* for continuous functions and *eraser game* for Baire class 1 functions, as well as Semmes's game for Baire class 2 functions [25], we get games for all the classes of the form $\mathbf{\Lambda}_{n,n}$, $\mathbf{\Lambda}_{n,n+1}$, and $\mathbf{\Lambda}_{n,n+2}$, respectively. The *multitape* and *multitape eraser* games of Semmes [24], which characterize $\mathbf{\Lambda}^{\mathsf{P}}_{3,3}$ and $\mathbf{\Lambda}^{\mathsf{P}}_{2,3}$ respectively — and thus also $\mathbf{\Lambda}_{3,3}$ and $\mathbf{\Lambda}_{2,3}$ by Theorem 2 — are particular cases of our construction.

3 Outline of future research and potential thesis

At this early stage, we envision a potential thesis divided into four chapters as detailed in the sections below. We anticipate submitting the results obtained in each part of the thesis to journals such as *Annals of Pure and Applied Logic*, *Fundamenta Mathematicae*, the *Journal of Symbolic Logic*, and the *Mathematical Logic Quarterly*, among other leading journals of mathematical logic.

Games for function classes and their applications

Although some other implicit, indirect, or reported but unpublished results exist [17, 15], only seven classes of functions of interest to us have known characterizations by what we consider *explicit* games (although at this moment we cannot make this distinction in an objective way): the classes $\mathbf{\Lambda}_{n,m}$ for $1 \leq n \leq m \leq 3$, and the class of all Borel functions. Therefore, there is plenty of work to be done in devising games characterizing the remaining $\mathbf{\Lambda}_{\alpha,\beta}$ functions, as well as classes of functions beyond Borel — where in particular questions of determinacy become relevant.

One promising idea is to use the complete choice principles F_X and R_X described in Section 2 to guide the definition of these games, in the spirit of how we can see the backtrack game as being *inspired* by C_ω as outlined at the end of Section 1.

Another approach to develop games which characterize new classes of functions is to develop a theory of operations on games which may allow us to obtain new games from old. Here the choice principles could also play a part; for example, one operation (called *parallelization*) is known in computable analysis, with the property that the parallelization of R_X is exactly F_X, at least in case X is Wadge-complete for some $\mathbf{\Sigma}^0_\alpha$. We are then interested in questions such as finding the game-theoretic counterpart of this operation.

As we described in Section 2, both the ideas of using complete choice principles and of defining operations on games have already started to bear fruit, but of course this is only the *tip of the iceberg*, as it were, and much remains to be done and understood.

Then, with new games defined we can try to generalize the ideas employed by Semmes in proving his two higher-level counterparts of the Jayne-Rogers theorem, and prove further generalizations of this result.

Let us stress that some results in this area have been claimed by Louveau [15], building on the work of Semmes, but that these results have never been published. Therefore, it would be desirable to contact Louveau and his students in order to discover the current state of the art of the subject.

Another good possibility for collaboration is Luca Motto Ros, in Torino, who has many interesting results in this area and has expressed interest in working together in our past meetings. This is an especially attractive prospect given the quality of the set theory group in Torino, which contains in particular Alessandro Andretta who also has done very relevant research in game-theoretic aspects of descriptive set theory.

Theory of choice principles

In order not only to apply choice principles and other tools and techniques from computable analysis to problems in descriptive set theory, but also to possibly uncover connections of computable analysis to other areas of mathematics, and finally for its own sake, it is highly desirable to further develop and understand the theory behind the choice principles. Some investigations in this direction have been done from the point of view of algebra [6, 11] and of category theory [22], but there is a lot of work to be done (see e.g. [21] for some open problems and directions of immediate research).

In particular, some of the applications of the choice principles C_k and C_ω rely on certain properties of these choice principles which we have not yet established for their respective generalizations F_X and R_X. During a recent research visit of mine to Arno Pauly in Cambridge this study was initiated, with many promising early results and indications of the directions of research in the immediate future. We are currently preparing an application for a grant from the Royal Society in order to fund multiple research visits between Amsterdam and Cambridge.

We would also like to mention the possibility of collaborating with Vasco Brattka, in Munich, who has done much of the foundational work in the part of computable analysis which interests us, and who was receptive to this idea in talks during the Colloquium Logicum, in September.

Variations of Weihrauch reducibility

In descriptive set theory, considerable attention has been given to variants of Wadge reducibility where one uses as reductions functions other than continuous ones (see, e.g., [1, 3, 16]). However, this kind of investigation has not been done for Weihrauch reducibility, although this is a very natural idea.

Another avenue for investigation along these lines is to restrict the functions used as reductions not (only) by their complexity, but (also) by the complexity of their domains (recall that in Weihrauch reducibility, the reductions IN and OUT are partial functions). In many cases, when one Weihrauch-reduces some relation R to S, although the reducing functions IN and OUT are continuous (w.r.t. the respective relative topologies of their domains), their domains are of high complexity as subsets of ω^ω. This allows a lot of of the complexity of the reduction as a whole to be *swept under the rug*, as it were. Therefore, it is to be expected that one could get finer control over these reductions by restricting their domains to some prescribed complexity.

Generalized settings

Although the idea of studying the descriptive set theory of the spaces κ^κ and 2^κ for cardinals $\kappa > \omega$ is not a recent one, it was only recently that this area started to receive an in-depth treatment (see [10] for an overview of the area and its history and applications).

However, many aspects of the theory of these spaces are not yet well-known. In particular, our main two areas of study, namely game characterizations of function classes and computable analysis, have not yet been investigated in the generalized case. We remark that doing computable analysis on these spaces would have to start with finding the appropriate model of transfinite computation among the existing alternatives such as infinite time Turing machines, ordinal Turing machines and infinite time register machines.

It is also worth mentioning that the 2014 edition of the *Amsterdam Workshop on Set Theory*, organized among others by me and my supervisor, will focus on the generalized Baire space. The aim is to organize what is currently known and compile a paper consisting of the open problems in the area.

Another generalized setting in which one can develop (some) descriptive set theory is in *quasi-Polish* spaces (as opposed to *Polish* spaces, of which ω^ω is the main representative).

These are particularly interesting for computer science because they include some non-Polish spaces such as ω-continuous domains. We refer the reader to [8] for an introduction to the theory of these spaces.

Since this area of research is also fairly recent, many aspects of the theory have still not been investigated, although some interesting results are known — e.g., it is known that the degree structure of subsets of a quasi-Polish space under Wadge reducibility can be *much* more poorly behaved than in the Polish case [19].

In particular, game characterizations of classes of functions are not known in this setting, and to the best of our knowledge not even the question of what the correct approach for computable analysis in quasi-Polish spaces is has been broached!

4 Research visits

Aside from attending the most relevant conferences and workshops in the area, I anticipate making multiple research visits during my PhD, although of course this will be done in a way compatible with my duties at the ILLC. Many of the possibilities for research visits were mentioned in the last section, but one concrete opportunity is the following.

From 19 August to 18 December 2015, the Isaac Newton Institute for Mathematical Sciences of the University of Cambridge will host the program titled *Mathematical, Foundational and Computational Aspects of the Higher Infinite*, whose stated goal is to "connect [the main] strands of set-theoretic research and other fields of set theory to the wider scope of mathematics, to research in the foundations of mathematics, including some philosophical issues, and to research on computational issues of infinity, e.g. in theoretical computer science and constructive mathematics". This program will have many leading and upcoming researchers in the different parts of set theory as participants, and will also include several conferences and workshops, among which we highlight the *5th European Set Theory Conference*. Full details of the program can be found at http://www.newton.ac.uk/event/hif.

I have had the honor of receiving a *visiting fellowship* for the whole duration of the program, including coverage of the cost of accommodation, as detailed in the invitation letter attached to this document. This participation will of course have an extremely positive influence on my research, and will furthermore allow for closer collaboration with Arno Pauly as well as facilitate new collaborations and networking with some of the most important researchers in set theory, both today and in years to come.

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